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INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

INTEGRAL SOLUTIONS OF THE BINARY QUADRATIC EQUATION
$$2x^2 - 4xy - y^2 + 20x - 2y + 17 = 0$$

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ABSTRACT

The hyperbola given by $2x^2 - 4xy - y^2 + 20x - 2y + 17 = 0$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented. Employing the solutions of the equation under consideration,the integer solutions for a few choices of hyperbola and parabola are obtained.

KEYWORDS: Binary Quadratic, Hyperbola, Parabola, Integral Solutions, Pell Equation 2010 Mathematics subject classification: 11D09.

INTRODUCTION

Diophantine equation of the form $y^2 = Dx^2 + 1$ where D is a given positive square free integer is known as pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L. Lagrange proved that the positive pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions where as the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a ceriterium for the solubility of the pell equation $x^2 - Dy^2 = -1$ where, D is any positive non-square integer, has been presented. For examples, the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solutions, where as $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-21].

In this communication, the hyperbola represented by $2x^2 - 4xy - y^2 + 20x - 2y + 17 = 0$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented.Employing the solutions of the equation under consideration,the integer solutions for a few choices of hyperbola and parabola are obtained.

METHODS FOR ANALYSIS

The binary quadratic equation representing hyperbola is

$$2x^2 - 4xy - y^2 + 20x - 2y + 17 = 0 \tag{1}$$

We present below two different patterns of non-zero distinct integer solutions of (1).

PATTERN:1

Treating (1) as a quadratic in x and solving for x,we have

$$x = -5 + y \pm \frac{\sqrt{6y^2 - 36y + 66}}{2} \tag{2}$$

Let $\alpha^2 = 6y^2 - 36y + 66$

$$\Rightarrow \alpha^2 = 6Y^2 + 12 \tag{3}$$

where $\quad Y = y - 3 \tag{3a}$

The initial solution of (3) is $\alpha_0 = 6, Y_0 = 2$

To find the other solution of (3),consider the pellian equation

$$\alpha^2 = 6Y^2 + 1 \tag{4}$$

whose general solution is given by

$$\tilde{\alpha}_s = \frac{1}{2} f_s, \tilde{Y}_s = \frac{1}{2\sqrt{6}} g_s$$

where

$$f_s = (5 + 2\sqrt{6})^{s+1} + (5 - 2\sqrt{6})^{s+1}$$
$$g_s = (5 + 2\sqrt{6})^{s+1} - (5 - 2\sqrt{6})^{s+1}, s = 0,1,2,3,\ldots\ldots$$

Applying Brahmagupta lemma between (α_0, Y_0) and $(\tilde{\alpha}_s, \tilde{Y}_s)$ the other integer solutions of (3) are given by

$$Y_{s+1} = f_s + \frac{3}{\sqrt{6}} g_s + 3$$

$$\alpha_{s+1} = 3f_s + \frac{6}{\sqrt{6}} g_s$$

Substituting the above values of α_{s+1}, Y_{s+1} in(3a),(2) and taking the positive sign on the RHS of(2),we get

$$x_{s+1} = \frac{5}{2} f_s + \sqrt{6} g_s - 2$$

$$y_{s+1} = f_s + \frac{\sqrt{6}}{2} + 3$$

which represent the integer solutions of(1).
The recurrence relations satisfied by x and y are given by

$$x_{s+1} + x_{s+3} - 10x_{s+2} = 16, x_0 = 3, x_1 = 47$$
$$y_{s+1} + y_{s+3} - 10y_{s+2} = -24, y_0 = 5, y_1 = 25$$

Some numerical examples of x and y satisfying (1) are given in the following table1.

Table 1: Numerical Examples

s	x_s	y_s
0	3	5
1	47	25
2	483	221
3	4799	2161
4	47523	21365
5	470447	211465

A few interesting properties between the solutions are given below:

1. $6(2x_{2s+2} - 4y_{2s+2} + 18)$ is a nasty number
2. $2x_{3s+3} - 4y_{3s+3} + 16 + 3(2_{s+1} - 4y_{s+1} + 16)$ is a cubic integer
3. $2x_{2s+2} - 4y_{2s+2} + 18$ is a perfect square
4. $y_{s+2} = y_{s+1} + 4x_{s+1} + 8$
5. $y_{s+3} = 29y_{s+1} + 20x_{s+1} + 56$
6. $2x_{s+2} = 4y_{s+1} + 18x_{s+1} + 20$
7. $2x_{s+3} = 40y_{s+1} + 178x_{s+1} + 232$

REMARKABLE OBSERVATIONS

Let p, q be two non-zero distinct positive integers such that $p = x_n + 2y_n, q = x_n$ note that $p > q > 0$. Treat

p, q as the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$ where $\alpha = 2pq$, $\beta = p^2 - q^2$ and $\gamma = p^2 + q^2$.
Let A,P represent the area and perimeter of T (α,β,γ). Then the following interesting relations are observed:

1. $\alpha - 27\beta + 26\gamma = 5$

2. $\dfrac{2A}{P} = x_{n+1}y_{n+1}$

3. By considering the linear combination among the solutions of (1), one may obtain solutions of different hyperbolas. A few examples are given in table2 below:

Table 2:Hyperbolas

S.No	x,y	Hyperbola
1	$2x_{s+1} - 4y_{s+1} + 16, -2x_{s+1} + 5y_{s+1} - 19$	$3x^2 - 2y^2 = 12$
2	$9y_{s+1} - y_{s+2} - 24, -2x_{s+1} + 5y_{s+1} - 19$	$54x^2 - 144y^2 = 864$
3	$89y_{s+1} - y_{s+3} - 264, -2x_{s+1} + 5y_{s+1} - 19$	$54x^2 - 14400y^2 = 86400$
4	$267y_{s+2} - 27y_{s+3} - 720, -2x_{s+1} + 5y_{s+1} - 19$	$x^2 - 24y^2 = 144$
5	$2x_{s+1} - 4y_{s+1} + 16, 11y_{s+1} - y_{s+2} - 30$	$6x^2 - y^2 = 24$
6	$2x_{s+1} - 4y_{s+1} + 16, 109y_{s+1} - y_{s+3} - 324$	$106x^2 - y^2 = 424$
7	$2x_{s+1} - 4y_{s+1} + 16, 109y_{s+2} - 11y_{s+3} - 294$	$36x^2 - y^2 = 144$
8	$10x_{s+1} - x_{s+2} + 18, -2x_{s+1} + 5y_{s+1} - 19$	$6x^2 - y^2 = 6$
9	$99x_{s+1} - x_{s+3} + 196, -2x_{s+1} + 5y_{s+1} - 19$	$54x^2 - 900y^2 = 5400$
10	$594x_{s+2} - 60x_{s+3} + 1068, -2x_{s+1} + 2y_{s+1} - 19$	$x^2 - 6y^2 = 36$
11	$2x_{s+1} - 4y_{s+1} + 16, 97x_{s+1} - x_{s+3} + 192$	$24x^2 - y^2 = 96$
12	$9y_{s+1} - y_{s+2} - 24, 11y_{s+1} - y_{s+2} - 30$	$3x^2 - 2y^2 = 48$
13	$267y_{s+2} - 27y_{s+3} - 720, 109y_{s+2} - 11y_{s+3} - 294$	$x^2 - 6y^2 = 144$
14	$89y_{s+1} - y_{s+3} - 264, 109y_{s+1} - y_{s+3} - 324$	$53x^2 - 200y^2 = 84800$
15	$10x_{s+1} - x_{s+2} + 18, 49x_{s+1} - 5x_{s+2} + 88$	$2x^2 - 3y^2 = 2$
16	$594x_{s+2} - 60x_{s+3} + 1068, 485x_{s+2} - 49x_{s+3} + 872$	$2x^2 - 3y^2 = 72$

By considering linear combination among the solutions of (1), one may obtain solutions of different parabolas. A few examples are given in table 3 below.

Table 3:Parabolas

S.No	x,y	Parabola
1	$2x_{2s+2} - 4y_{2s+2} + 18, 11y_{s+1} - y_{s+2} - 30$	$y^2 = 6x - 24$
2	$2x_{2s+2} - 4y_{2s+2} + 18, 109y_{s+2} - y_{s+3} - 324$	$y^2 = 106x - 424$
3	$2x_{2s+2} - 4y_{2s+2} + 18, 49x_{s+1} - 5x_{s+2} + 88$	$6y^2 = x - 4$
4	$2x_{2s+2} - 4y_{2s+2} + 18, 97x_{s+1} - x_{s+3} + 192$	$y^2 = 24x - 96$
5	$2x_{2s+2} - 4y_{2s+2} + 18, -2x_{s+1} + 5y_{s+1} - 19$	$2y^2 = 3x - 12$

Note :By considering the negative sign on the RHS of (2),another set of x-values for the same y-values are obtained.

PATTERN: 2
Treating (1) as a quadratic in y and solving for y, we have

$$y = -1 - 2x \pm \sqrt{6(x^2 + 4x + 3)} \qquad (5)$$

$$\text{Let } \beta^2 = 6(x^2 + 4x + 3)$$

$$\Rightarrow \beta^2 = 6X^2 - 6 \qquad (6)$$

where $\quad X = x + 2 \qquad (6a)$

The initial solution of (6) is $\beta_0 = 0, X_0 = 1$

To find the other solution of (6), consider the Pellian equation

$$\beta^2 = 6X^2 + 1 \qquad (7)$$

whose general solution is given by

$$\tilde{\beta}_s = \frac{1}{2}f_s, \tilde{X}_s = \frac{1}{2\sqrt{6}}g_s$$

where

$$f_s = (5 + 2\sqrt{6})^{s+1} + (5 - 2\sqrt{6})^{s+1}$$

$$g_s = (5 + 2\sqrt{6})^{s+1} + (5 - 2\sqrt{6})^{s+1}, \text{ s=0,1,2,3.............}$$

Applying Brahmagupta lemma between (X_0, β_0) and ($\tilde{X}_s, \tilde{\beta}_s$) the other integer solutions of (6) are given by

$$X_{s+1} = \frac{1}{2}f_s$$

$$\beta_{s+1} = \frac{\sqrt{6}}{2}g_s$$

Substituting the above values of X_{s+1}, β_{s+1} in(6a),(5) and taking the positive sign on the RHS of(5), we get

$$x_{s+1} = \frac{1}{2}f_s - 2$$

$$y_{s+1} = -f_s + \frac{\sqrt{6}}{2}g_s + 3$$

The recurrence relations satisfied by x and y are given by

$$x_{s+1} + x_{s+3} - 10x_{s+2} = 16, \quad x_0 = -1, x_1 = 3$$

$$y_{s+1} + y_{s+3} - 10y_{s+2} = -24, \ Y_0 = 1, Y_1 = 5$$

Same numerical examples of x and y satisfying (1) are given in the following table (4)

Table 4: Numerical Examples

s	x_s	y_s
0	-1	1
1	3	1
2	47	25
3	483	221
4	4799	2161
5	47523	21365

A few interesting properties between the solutions are given below

1. $6(2x_{2s+2} + 6)$ is a nasty number

2. $2x_{2s+2} + 6$ is a perfect square

3. $2x_{3s+3} + 6x_{s+1} + 16$ is a cubic integer

4. $y_{s+2} = 4x_{s+1} + y_{s+1} + 8$

5. $y_{s+3} = 40x_{s+1} + 9y_{s+1} + 56$

6. $x_{s+2} = 9x_{s+1} + 2y_{s+1} + 10$

7. $x_{s+3} = 89x_{s+1} + 20y_{s+1} + 116$

By considering linear combination among the solutions of (1), one may obtain solutions of different hyperbolas. A few examples are given in table (5) below

Table 5:Hyperbolas

S.No	x,y	Hyperbola
1	$2x_{s+1} + 4, 2x_{s+1} + y_{s+1} + 1$	$3x^2 - 2y^2 = 8$
2	$2x_{s+1} + 4, 5x_{s+1} - x_{s+2} + 8$	$6x^2 - y^2 = 24$
3	$2x_{s+1} + 4, 49x_{s+1} - x_{s+3} + 96$	$106x^2 - y^2 = 424$
4	$10x_{s+1} - x_{s+3} + 18, 2x_{s+1} + y_{s+1} + 1$	$6x^2 - y^2 = 6$
5	$y_{s+1} - y_{s+2}, 2x_{s+1} + y_{s+1} + 1$	$3x^2 - 2y^2 = 48$
6	$9y_{s+1} - y_{s+3} - 24, 2x_{s+1} + y_{s+1} + 1$	$3x^2 - 800y^2 = 4800$
7	$9y_{s+2} - y_{s+3} - 24, 2x_{s+1} + y_{s+1} + 1$	$3x^2 - 8y^2 = 48$
8	$10x_{s+2} - x_{s+3} + 18, 49x_{s+2} - 5x_{s+3} + 88$	$24x^2 - y^2 = 24$
9	$9y_{s+1} - y_{s+3} - 24, 11y_{s+1} + y_{s+3} - 36$	$53x^2 - 200y^2 = 84800$

By considering linear combination among the solutions of (1), one may obtain solutions of different parabolas. A few examples are given in table 6 below.

Table 6:Parabolas

S.No	x,y	Parabola
1	$2x_{2s+2} + 6, \ 2x_{s+1} + y_{s+1} + 1$	$2y^2 = 3x - 12$
2	$2x_{2s+2} + 6, \ 5x_{s+1} - x_{s+2} + 8$	$y^2 = 6x - 24$
3	$2x_{2s+2} + 6, \ 49x_{s+1} - x_{s+3} + 96$	$y^2 = 106x - 424$

Note :By considering the negative sign on the RHS of (5),another set of x-values for the same y-values are obtained.

CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by $2x^2 - 4xy - y^2 + 20x - 2y + 17 = 0$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of hyperbolic and determine their integer solutions along with suitable properties.

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